3. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Higher Analysis [in Russian], Fizmatgiz, Moscow (1962).
4. I. S. Berezin and N. P. Zhidkov, Calculation Methods [in Russian], Vol. 2, Fizmatgiz, Moscow (1962).
5. V. L. Rvachev and A. P. Slesarenko, The Algebra of Logic and Integral Transforms in Boundary-Value Problems [in Russian], Naukova Dumka, Kiev (1976).
6. P. V. Tsoi, Methods of Solving Individual Problems of Heat and Mass Transfer [in Russian], Énergiya, Moscow (1971).

NONSTEADY HEAT TRANSFER FROM A CYLINDER WITH INJECTION

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The problem of nonsteady heat transfer from a cylinder in the presence of radial injection is analyzed. The heat flux at the surface of the cylinder is found from the assigned variation in the surface temperature by the method of [1].

The nonsteady heat transfer from a cylinder to an infinite external medium in the presence of radial injection is described by the problem

$$\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\rho^2}-\frac{1-\mathrm{Pe}}{\rho}\cdot\frac{\partial}{\partial\rho}\right)T=0,$$
$$1\leqslant\rho<\infty,\quad 0<\tau<\infty, \tag{1}$$
$$T|_{\rho=1}=T_1(\tau);\quad T|_{\rho=\infty}=0;\quad T|_{\tau=0}=0,$$
$$\tau=R^2t/a;\quad \rho=r/R;\quad \mathrm{Pe}=uR/a.$$

The temperature gradient $q_1 = (\partial T/\partial\rho)_{\rho=1}$ at the surface of the cylinder is to be determined.

Despite the apparent simplicity of the problem, a solution by the operational method is very laborious, since the Laplace transform of the solution has the form

$$\bar{q}_1=\mathrm{Pe}/2+\sqrt{p}\,K'_{\mathrm{Pe}/2}(\sqrt{p})\,K^{-1}_{\mathrm{Pe}/2}(\sqrt{p})$$

(K is the MacDonald function and p is the parameter of the Laplace transform), and the primitive function is expressed through a complicated integral of special functions, even for the simplest case of Pe = 0 (see [2]).

Therefore, we carry out the solution following the method presented in [1], where a similar problem is analyzed for the equation

$$\left[\frac{\partial}{\partial\tau}-\alpha(\rho,\ \tau)\frac{\partial^2}{\partial\rho^2}-\beta(\rho,\ \tau)\frac{\partial}{\partial\rho}+\gamma(\rho,\ \tau)\right]T=0 \tag{2}$$

(the difference is only in the notation).

The solution of the problem (1) is expressed in the form of a series with respect to the derivative of the half-integral index of the assigned surface temperature $T_1(\tau)$:

$$-q_1(\tau)=\sum_{n=0}^{\infty}a_n(1)\,D^{\frac{1-n}{2}}\,T_1(\tau). \tag{3}$$

Here the operator for the fractional derivative of order ν is defined by the expression

$$D^{\nu}T_1(\tau)=\frac{1}{\Gamma(1-\nu)}\cdot\frac{d}{d\tau}\int_0^{\tau}(\tau-z)^{-\nu}\,T_1(z)\,dz,\quad \nu<1, \tag{4}$$

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while the quantities $a_n(1) = a_n(\rho)_{\rho=1}$ are found from the system of recurrent equations

$$\begin{gathered}
a_0 = 1, \quad a_1 = (1 - \mathrm{Pe})/2\rho, \\
2a_2 = (\mathrm{Pe}^2 - 1)/4\rho^2, \\
2a_3 = a_2', \\
2a_4 = a_3' - a_2^2, \\
2a_5 = a_4' - a_3a_2 - a_2a_3, \\
\cdots\cdots\cdots\cdots \\
2a_n = a_{n-1}' - \sum_{k=0}^{n-4} a_{n-2-k}a_{2+k} \\
\cdots\cdots\cdots\cdots
\end{gathered} \tag{5}$$

The prime denotes a derivative with respect to ρ.

The system (5) can be reduced to the algebraic substitution $a_n = A_n\rho^{-n}$. For the determination of $A_n = a_n$ of (1) we obtain the system

$$\begin{gathered}
A_0 = 1, \quad A_1 = (1 - \mathrm{Pe})/2, \\
2A_2 = (\mathrm{Pe}^2 - 1)/4, \\
2A_3 = -2A_2, \\
2A_4 = -3A_3 - A_2^2, \\
2A_5 = -4A_4 - A_3A_2 - A_2A_3, \\
\cdots\cdots\cdots\cdots \\
2A_n = -(n-1)A_{n-1} - \sum_{k=0}^{n-4} A_{n-2-k}A_{2+k}.
\end{gathered} \tag{6}$$

The solution of the stated problem is given by Eqs. (3), (4), and (6).

Let us investigate the convergence of the series (3). It should be noted that the problem (1) is "unfavorable" for the use of the method of [1], since the successive derivatives of the function $(1 - \mathrm{Pe})/2\rho$ grow too rapidly. In this case the sufficient condition for convergence of the series (3) obtained in [3] is not satisfied.

Remark. In order to use the condition of [3], one must first convert Eq. (1) into the equation of [3] using the obvious substitutions, where one must set $\gamma = (\mathrm{Pe}^2 - 1)/4\rho$.

Nevertheless, we can see that in many cases (3) is an absolutely converging series.

From the definition (4), using the mean-value theorem, one can obtain the estimate

$$D^{\frac{1-n}{2}} T_1(\tau) \leqslant \Gamma^{-1}\left(\frac{n+1}{2}\right) \tau^{\frac{n-1}{2}} \sup T_1, \quad n \geqslant 1. \tag{7}$$

Consequently, the absolute convergence of the series (3) for all τ requires the condition

$$A_n \sim o\left[\Gamma\left(\frac{n+1}{2}\right)\right]. \tag{8}$$

If $\mathrm{Pe}^2 < 1$ then all the terms on the right sides of Eqs. (6) have the same signs. It is seen that A_n grow faster than $\Gamma(n)$, the condition (8) is not satisfied, and the series (3) diverges for all $\tau > 0$. One can show that it is an asymptotic series and is suitable for calculations with small enough τ. For Pe = 0, for example,

$$-q_1(\tau) = \left(D^{1/2} + \frac{1}{2} - \frac{1}{8} D^{-1/2} + \frac{1}{8} D^{-1} - \frac{25}{128} D^{-3/2} + \right.$$
$$\left. + \frac{13}{32} D^{-2} - \frac{1073}{1024} D^{-5/2} + \frac{103}{32} D^{-3} - \ldots\right) T_1(\tau).$$

Four terms of this series for T_1 = const are presented in [2].

The error of the calculation does not exceed the first discarded term in absolute value and has the sign of the last retained term.

If $Pe^2 > 1$, then the signs of the individual terms on the right sides of Eqs. (6) are different and the investigation of the behavior of A_n is more complicated. We introduce the generating function

$$S = \sum_{k=0}^{\infty} A_{2+k} z^k \tag{9}$$

into the analysis. We multiply the line of the system (6) defining A_3 by z, the line defining A_4 by z^2, etc. Adding the lines (starting with A_2), we obtain an equation determining S:

$$\begin{gathered}\frac{dS}{dz} = S^2 + \left(\frac{2}{z} - \frac{2}{z^2}\right) S + \frac{N^2}{z^2}, \\ S|_{z=0} = N/2; \quad N = (\mathrm{Pe}^2 - 1)/4.\end{gathered} \tag{10}$$

The solution of (10) can be expressed through the solution of the second-order linear equation

$$S = -\frac{1}{u} \cdot \frac{du}{dz}, \quad z^2 u'' + 2(1-z) u' + Nu = 0, \tag{11}$$

where the function u is represented by the power series

$$\begin{gathered}u = \sum_{n=0}^{\infty} b_n z^n, \quad b_0 = 1, \quad b_1 = -N/2, \\ b_{n+1} = \frac{n(n-1) - N}{2(n+1)} b_n.\end{gathered} \tag{12}$$

In the general case the series (12) diverges. For $N = n(n-1)$ (n is an integer) the series terminates and the solution of Eq. (10) has the form of a linear-fractional function, from which it follows that the series (9) has a finite radius of convergence. Therefore, the condition (8) is satisfied and the series (3) converges absolutely for all $\tau > 0$ for the indicated N. This corresponds to $Pe = 2k + 1$, $\pm k = 0, 1, 2, \ldots$. Thus, the absolute convergence of the series (3) is established for all odd values of $|Pe|$.

The question of the convergence for the general case of $Pe^2 > 1$ remains open.

Let us write the solution for some particular cases, using Eq. (4) to perform the summation:

$$\mathrm{Pe} = 1; \quad -q_1(\tau) = D^{1/2} T_1(\tau),$$

$$\mathrm{Pe} = -1; \quad -q_1(\tau) = (D^{1/2} + 1) T_1(\tau),$$

$$\mathrm{Pe} = 3; \quad -q_1(\tau) = \sum_{n=0}^{\infty} (-1)^n D^{\frac{1-n}{2}} T_1(\tau) = \frac{d}{d\tau} \int_0^{\tau} \left[D^{1/2} T_1(z) - T_1(z)\right] e^{\tau - z} dz,$$

$$\begin{aligned}\mathrm{Pe} = 5; \quad -q_1(\tau) &= (D^{1/2} - 2 + 3D^{-1/2} - 3D^{-1}) T_1(\tau) + (9D^{-2} - 27D^{-5/2} + \\ &+ 54D^{-3} - 81D^{-7/2} + 81D^{-4}) \sum_{n=0}^{\infty} (-1)^n 3^{3n} D^{-3n} T_1(\tau) = \\ &= (D^{1/2} - 2 + 3D^{-1/2} - 3D^{-1}) T_1(\tau) + (3D^{-2} - 9D^{-5/2} + 18D^{-3} - \\ &- 27D^{-7/2} + 27D^{-4}) \frac{d}{d\tau} \int_0^{\tau} \left[e^{-3(\tau - z)} + 2e^{3(\tau - z)/2} \cos \frac{3\sqrt{3}}{2} (\tau - z)\right] T_1(z)\, dz.\end{aligned}$$

Let us obtain the asymptotic form of Eq. (3) as $|Pe| \to \infty$. For this we retain the terms with the leading power of N in each line of the system (8). Then (3) is rewritten in the form

$$-q_1(\tau) = -\frac{\mathrm{Pe}}{2}\, T_1(\tau) + \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} N^n D^{\frac{1}{2}-n} T_1(\tau) +$$

$$+\frac{1}{2}\sum_{n=0}^{\infty} (-1)^n N^n D^{-n} T_1(\tau) = -\frac{\mathrm{Pe}}{2} T_1(\tau) + \exp(-\mathrm{Pe}^2\tau/4)\, D^{1/2} \times$$

$$\times [\exp(\mathrm{Pe}^2\tau/4)\, T_1(\tau)] + \frac{1}{2}\frac{d}{d\tau}\int_0^{\tau} \exp[-\mathrm{Pe}^2(\tau - z)/4]\, T_1(z)\, dz.$$

From this one can obtain the well-known steady solution as $\tau \to \infty$:

$$-q_1(\infty) = \left(\sqrt{\frac{\mathrm{Pe}^2}{4}} - \frac{\mathrm{Pe}}{2}\right) T_1(\infty).$$

The problem analyzed seems important to us in a mathematical respect for the following reason. It was shown that finding the terms of the series (3) comes down to the performance of the algebraic operations (6) only. The solution can be reduced to algebraic operations similarly in the case when α = const while β and γ are finite polynomials of the functions ρ, ρ^{-1}, and exp (const$\cdot\rho$) in the general equation (2).

NOTATION

A_n, a_n, constants and functions entering into the solution; a, thermal diffusivity; b_n, coefficient in the solution of the auxiliary equation; D^ν, fractional differentiation sign; K, MacDonald function; N, function of Pe; Pe, Peclet number; R, radius of cylinder; p, parameter of the Laplace transform; q_1, temperature gradient at surface of cylinder; r, radial coordinate; S, generating function; T, temperature; T_1, surface temperature of cylinder; t, time; u, velocity of liquid stream at surface of cylinder; z, variable in generating function and integration variable; α, β, γ, coefficients of general equation of heat conduction; ρ, dimensionless radial coordinate; τ, dimensionless time; n, k, ν, indices and exponents.

LITERATURE CITED

1. Yu. I. Babenko, "Use of the fractional derivative in problems of heat-transfer theory," in: Heat and Mass Transfer [in Russian], Vol. 8, Minsk (1972), p. 541.
2. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 2nd ed., Clarendon Press, Oxford (1959).
3. Yu. I. Babenko, "Heat transfer to an unevenly cooled rod," Inzh.-Fiz. Zh., 26, No. 3 (1974).